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Introducing a Geometric Potential Theory for two-dimensional steady flows

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Abstract It is demonstrated that the continuity and irrotationality conditions imply geometric constraints for two-dimensional steady flows. This fact leads to the formulation of an exclusively geometric criterion for these kinematical conditions. Thus, flow visualization can be conclusive on whether a flow possesses a potential or not. Moreover, the mutual relationship among the flow geometry, its kinematics and physics, can be expressed mathematically using an eikonal equation. Its analytical solution in the flow-streamline coordinate system enables the formulation of the *'Geometric Potential Theory'*. Accordingly, the determination of the physical quantities of velocity and static pressure throughout the flow is reduced to the purely geometrical problem of finding the streamline and potential line curvatures. These two functions are combined to introduce the *'Global Curvature Vector'*, a vector that can be mathematically and physically interpreted. Finally, it is shown that continuity and irrotationality are identically satisfied by the existence of a *'Curvature Potential'*, that is, the existence of an analytic expression from which the global curvature vector components can be found by partial differentiation.

Keywords Curvature potential · Eikonal equation · Geometric potential theory · Global curvature vector · Potential flow

1 Introduction

In potential theory a flow-field velocity which satisfies the continuity and irrotationality conditions can be expressed in terms of a potential function Φ, such that $\boldsymbol{v} = \nabla\Phi$. Finding this scalar for a specific flow requires solving the Laplace equation

$$\nabla^2\Phi = 0. \tag{1.1}$$

It turns out that another approach can be employed for the calculation of the velocity field. Similar to the existing potential theory, velocity could be expressed in terms of a single variable that has a purely geometric content. This consideration is inspired by the fact that the potential function Φ contains the geometric information of the flow that it describes, since its isolines are always perpendicular to the flow streamlines. Initially in this argument it was thought that the potential function, Φ, could be replaced by a scalar which involves the flow-field geometry in

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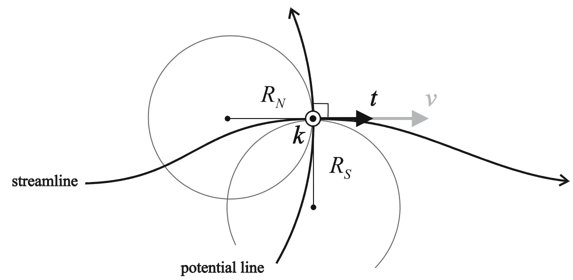


Fig. 1 Principal radii of curvature, $R_S = K_S^{-1}$ and $R_N = K_N^{-1}$

terms of the streamline curvature, K_S. A closer investigation, however, illustrates that knowledge of the streamline curvature alone is not sufficient to replace the potential function Φ [1]. Instead, a combination of the streamline and potential line curvatures, K_S and K_N respectively, with

$$K_S = |\nabla \times \boldsymbol{t}| \text{ or in vector form } \boldsymbol{K_S} = \nabla \times \boldsymbol{t} = |\nabla \times \boldsymbol{t}|\,\boldsymbol{k} \text{ and} \tag{1.2}$$

$$K_N = \nabla \cdot \boldsymbol{t} \text{ or in vector form } \boldsymbol{K_N} = (\nabla \cdot \boldsymbol{t})\,\boldsymbol{k}, \tag{1.3}$$

where $\boldsymbol{t} = \frac{\boldsymbol{v}}{v}$ defines the normalized or direction vector field[1] and $\boldsymbol{k}$ the unit vector perpendicular to the plane of the fluid motion, proves to be the means necessary for representing the new potential function (Fig. 1). Since every point in the flow field can be geometrically characterized in terms of curvature, from both the stream and potential lines that pass through it, it is therefore logical to define a global quantity comprised of both curvatures. This quantity is the Global Curvature (K_G) [1] and is defined as

$$K_G^2 = K_S^2 + K_N^2. \tag{1.4}$$

The mutual relation between the geometry and the kinematics of a flow, which satisfies the continuity ($\nabla \cdot \boldsymbol{v} = 0$) and irrotationality conditions ($\nabla \times \boldsymbol{v} = \boldsymbol{0}$), can be demonstrated by applying the following identities

$$\nabla \cdot (f\boldsymbol{F}) = f\nabla \cdot \boldsymbol{F} + \nabla f \cdot \boldsymbol{F} \quad \text{and} \quad \nabla \times (f\boldsymbol{F}) = f\nabla \times \boldsymbol{F} + \nabla f \times \boldsymbol{F},$$

to the velocity vector field $\boldsymbol{v}$. This leads to the subsequent equations

$$0 = v\nabla \cdot \boldsymbol{t} + \nabla v \cdot \boldsymbol{t} \overset{(1.3)}{\Leftrightarrow} vK_N = -\nabla v \cdot \boldsymbol{t} \text{ and} \tag{1.5}$$

$$\boldsymbol{0} = v\nabla \times \boldsymbol{t} + \nabla v \times \boldsymbol{t} \overset{(1.2)}{\Leftrightarrow} vK_S\boldsymbol{k} = \boldsymbol{t} \times \nabla v, \tag{1.6}$$

where $\boldsymbol{k}$ is the unit vector perpendicular to the plane of the fluid motion pointing out of the page (Fig. 1). If now the last equations are squared and then added together, they finally give

$$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = v^2 K_G^2 \text{ or } \boxed{|\nabla v| = vK_G}, \tag{1.7}$$

as shown in [1]. Therefore, the efforts to link K_G with the kinematics of a potential flow, lead to a partial differential equation with velocity being the only unknown quantity. Furthermore, the physics of such a flow (continuity and irrotationality satisfied) can also be related to its geometry by using the Bernoulli equation

$$P + \frac{1}{2}\rho v^2 = P_{\text{tot}} = \text{const}, \tag{1.8}$$

where P_{tot} is the sum of the static pressure P and the dynamic pressure $(1/2)\,\rho v^2$, also called the Bernoulli constant. By substituting Eq. 1.8 in Eq. 1.7, the desired relation is obtained:

$$(1.7) \overset{(1.8)}{\Rightarrow} \left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2 = 4(P_{\text{tot}} - P)^2 K_G^2 \quad \text{or} \quad \boxed{|\nabla P| = 2\,(P_{\text{tot}} - P)\,K_G}.$$

[1] This definition and consequently Eqs. 1.2 and 1.3 are valid for non-critical points only.

The nonlinear partial differential Eq. 1.7 can be simplified by introducing a new variable $N = N(x, y)$, which is defined for non-critical points as the natural logarithm of the velocity magnitude,

$$N = \log v \quad \text{with } v > 0. \tag{1.9}$$

Finally with the aid of Eq. 1.9 we obtain

$$\left(\frac{\partial N}{\partial x}\right)^2 + \left(\frac{\partial N}{\partial y}\right)^2 = K_G^2 \quad \text{or} \quad \boxed{|\nabla N| = K_G}. \tag{1.10}$$

Equation (1.10), also called the eikonal equation, offers a novel approach to determine the flow-velocity distribution, while satisfying the continuity and irrotationality conditions. Direct knowledge of the global curvature function can then lead to the determination of the velocity field properties.

Solving Eq. 1.10 in case of a known global curvature function is not trivial and requires the use of numerical methods. Nevertheless, it will be demonstrated that an analytical expression for the velocity magnitude, v, can be obtained as a function of the intrinsic coordinates (s, n), where s and n are the distances along and normal to the streamlines, respectively. Moreover, a geometric criterion is obtained which allows for the determination of whether a Laplacian potential flow is represented by inspecting a flow visualization. The developed ideas are then applied to three elementary flows to obtain their respective velocity distributions. Finally, the formulation of the Geometric Potential theory is completed by giving a 'vectorial content' to the flow field's Global Curvature[2] and introducing the 'Method of Curvature Potential' for the computation of its components.

2 Analytical solution of the eikonal equation

To solve Eq. 1.10, it is necessary to express it in the intrinsic coordinate system (s, n). First, a velocity vector $\boldsymbol{v}$ originating at point P is considered (Fig. 2). The unit vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ are tangent to the streamlines and potential lines, respectively. Their directions vary from point to point but remain perpendicular to each other, thus forming an orthogonal coordinate system (s, n). The gradient of the velocity magnitude can be expressed both in the Cartesian and streamline coordinates as follows:

$$\nabla v = \left(\frac{\partial v}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial v}{\partial y}\right)\boldsymbol{j} \quad \text{in Cartesian coordinates,} \tag{2.1}$$

$$\nabla v = \frac{1}{\left|\frac{\partial \boldsymbol{r}}{\partial s}\right|}\left(\frac{\partial v}{\partial s}\right)\boldsymbol{t} + \frac{1}{\left|\frac{\partial \boldsymbol{r}}{\partial n}\right|}\left(\frac{\partial v}{\partial n}\right)\boldsymbol{n} \quad \text{in streamline coordinates,} \tag{2.2}$$

where the term $\mathrm{d}\boldsymbol{r}$ in Eq. 2.2 denotes an infinitesimal increment of the position vector $\boldsymbol{r}$ of P in the directions of s and n. Since now the parameters s and n are the distances along the streamlines and potential lines, respectively, it holds that

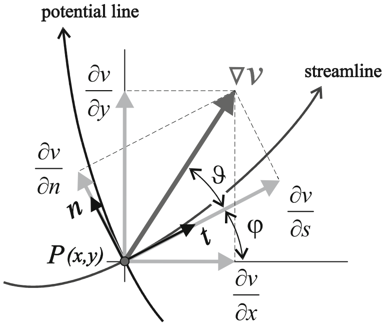


Fig. 2 Components of the velocity gradient ∇v, in Cartesian and streamline coordinate systems

[2] The Global Curvature is defined up to now as a scalar quantity [1].

$$\left|\frac{\partial \boldsymbol{r}}{\partial s}\right| = \left|\frac{\partial \boldsymbol{r}}{\partial n}\right| = 1. \tag{2.3}$$

This is a well-known result from classical differential geometry according to which the 'velocity' of a curve, which is parameterized with its arclength, is constant and equal to 1 [2]. Therefore, Eq. 2.2 can be written as

$$\nabla \nu = \left(\frac{\partial \nu}{\partial s}\right)\boldsymbol{t} + \left(\frac{\partial \nu}{\partial n}\right)\boldsymbol{n}. \tag{2.4}$$

By substituting the previous mathematical expression, for the velocity gradient in streamline coordinates, in Eq. 1.5 and Eq. 1.6, we obtain:

$$(1.5) \overset{(2.4)}{\Rightarrow} \nu K_N = -\left(\left(\frac{\partial \nu}{\partial s}\right)\boldsymbol{t} + \left(\frac{\partial \nu}{\partial n}\right)\boldsymbol{n}\right)\cdot \boldsymbol{t} \;\Leftrightarrow\; \nu K_N = -\frac{\partial \nu}{\partial s}, \tag{2.5}$$

$$(1.6) \overset{(2.4)}{\Rightarrow} \nu K_S \boldsymbol{k} = \boldsymbol{t} \times \left(\left(\frac{\partial \nu}{\partial s}\right)\boldsymbol{t} + \left(\frac{\partial \nu}{\partial n}\right)\boldsymbol{n}\right) \;\Leftrightarrow\; \nu K_S = \frac{\partial \nu}{\partial n}. \tag{2.6}$$

Subsequently squaring and adding the previous two equations yields:

$$|\nabla N| = K_G \quad \text{with } N = \log\nu \; (\nu > 0)\,, \tag{2.7}$$

where the quantities N and K_G are functions of (s, n). The eikonal equation is obtained in the same form as for Cartesian coordinates in Eq. 1.10. Since the global curvature has a purely geometric content, it is an invariant quantity, thus leaving the form of Eq. 2.7 unaffected under coordinate transformations.

Equations (2.5) and (2.6) are linear, first-order differential equations with the velocity magnitude being the unknown. They can be rewritten in the form:

$$K_N = -\frac{1}{\nu}\frac{\partial \nu}{\partial s} \tag{2.8}$$

$$K_S = \frac{1}{\nu}\frac{\partial \nu}{\partial n}. \tag{2.9}$$

This formulation clearly illustrates their physical meaning. *The potential line curvature, K_N, at a given point $\boldsymbol{P}$ is the negative*[3] *percentage rate of change of the velocity magnitude, ν, in the streamwise direction at that point* (Eq. 2.8). *The streamline curvature, K_S, equals the percentage rate of change of ν in the normal direction* (Eq. 2.9).

Solving this system of equations results in the velocity magnitude at every point in the potential flow as a function of the streamline coordinates (s, n). To demonstrate this, Eqs. 2.8 and 2.9 are rewritten as follows

$$(2.8) \Rightarrow K_N = -\frac{\partial \log \nu}{\partial s} \tag{2.10}$$

and

$$(2.9) \Rightarrow K_S = \frac{\partial \log \nu}{\partial n}. \tag{2.11}$$

Upon integration with respect to s and n, respectively, the following equations are obtained:

$$(2.10) \Rightarrow \int K_N \,\mathrm{d}s - F(n) = -\int \frac{\partial \log \nu}{\partial s}\,\mathrm{d}s \Rightarrow \log \nu = -\int K_N \,\mathrm{d}s + F(n), \tag{2.12}$$

$$(2.11) \Rightarrow \int K_S \,\mathrm{d}n + G(s) = \int \frac{\partial \log \nu}{\partial n}\,\mathrm{d}n \Rightarrow \log \nu = \int K_S \,\mathrm{d}n + G(s), \tag{2.13}$$

where F and G depend only on n and s. Therefore,

$$\nu = \exp\left(-\int K_N \,\mathrm{d}s + F(n)\right) \tag{2.14}$$

and

[3] In Eqs. 2.8 and 2.9 the two curvature functions can take negative values as well, according to the right-hand rule. The 'signed' curvature is considered rather than its absolute value.

$$\nu = \exp\left(\int K_S \,\mathrm{d}n + G(s)\right). \tag{2.15}$$

The terms $F(n)$ and $G(s)$ are generally unknown quantities. However, they can be found as long as K_S and K_N are known. The equality between Eqs. 2.12 and 2.13 implies that:

$$G(s) = -\int K_N \,\mathrm{d}s - \int K_S \,\mathrm{d}n + F(n). \tag{2.16}$$

If $F(n)$ is known, $G(s)$ can be determined from Eq. 2.16. To find $F(n)$, Eq. 2.16 is differentiated with respect to n:

$$\begin{aligned}
&\frac{\partial G(s)}{\partial n} = 0 \Rightarrow \frac{\partial}{\partial n}\left[-\int K_N \,\mathrm{d}s - \int K_S \,\mathrm{d}n + F(n)\right] = 0 \Rightarrow \\
&-\frac{\partial}{\partial n}\int K_N \,\mathrm{d}s - K_S + \frac{\mathrm{d}F(n)}{\mathrm{d}n} = 0 \Rightarrow \frac{\mathrm{d}F(n)}{\mathrm{d}n} = K_S + \frac{\partial}{\partial n}\int K_N \,\mathrm{d}s \Rightarrow \\
&F(n) = \int K_S \,\mathrm{d}n + \int\int \frac{\partial K_N}{\partial n}\,\mathrm{d}s\mathrm{d}n + c.
\end{aligned} \tag{2.17}$$

Consequently, by combining Eqs. 2.14 and 2.17:

$$\begin{aligned}
&\nu = \exp\left(-\int K_N \,\mathrm{d}s + \int K_S \,\mathrm{d}n + \iint \frac{\partial K_N}{\partial n}\,\mathrm{d}s\mathrm{d}n + c\right) \Rightarrow \\
&\boxed{\nu = C\exp\left(\int K_S \,\mathrm{d}n - \int K_N \,\mathrm{d}s + \iint \frac{\partial K_N}{\partial n}\,\mathrm{d}s\mathrm{d}n\right) \text{ where } C = \mathrm{e}^c = \text{const}}.
\end{aligned} \tag{2.18}$$

This equation provides the velocity magnitude ν if K_S and K_N are specified functions in terms of the streamline coordinates and the constant C is determined from the boundary conditions of the problem.

To complete the computation of the velocity vector field, its direction $\boldsymbol{t}$ must also be resolved. This can theoretically be performed given its divergence and rotation (*Curl*). This is a known mathematical problem, common in the areas of electrodynamics and plasma physics. In general, for the following differential equation,

$$\nabla \times \boldsymbol{\nu} = 4\pi \boldsymbol{J}, \tag{2.19}$$

where $\boldsymbol{\nu}$ is sought while $\boldsymbol{J}$ is known, to have a unique solution there are two conditions that must be satisfied [3]. The first condition is that the divergence of $\boldsymbol{\nu}$,

$$\nabla \cdot \boldsymbol{\nu} = 4\pi\lambda, \tag{2.20}$$

must be known. This means that the scalar λ, which satisfies Eq. 2.20, must be known. The second condition requires that the divergence of the vector $\boldsymbol{J}$ vanishes,

$$\nabla \cdot \boldsymbol{J} = 0, \tag{2.21}$$

since only then the initial differential Eq. 2.19 is valid, according to the fact that the divergence of the rotation of any vector field vanishes.

In the presented case, the divergence and rotation of the normalized velocity vector can be used to determine it at every point in the flow field. The relationship between $\boldsymbol{t}$ and $\boldsymbol{K_S}$ is shown in Eq. 1.2 and has been developed in [1]:

$$\nabla \times \boldsymbol{t} = \boldsymbol{K_S}. \tag{2.22}$$

Its similarity to Eq. 2.19 leads to the assumption that $\boldsymbol{t}$ can be solved, given its divergence and rotation. According to Eq. 1.3,

$$\nabla \cdot \boldsymbol{t} = K_N, \tag{2.23}$$

which implies that the potential line curvature, K_N, must be known and again shows the similarity to Eq. 2.20. The second condition requires that the divergence of $\boldsymbol{K_S}$ vanishes. In Cartesian coordinates it is:

$$\nabla \cdot \boldsymbol{K_S} \overset{(1.2)}{=} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (0, 0, K_S(x, y)) = \frac{\partial K_S(x, y)}{\partial z} \Rightarrow \nabla \cdot \boldsymbol{K_S} = 0. \tag{2.24}$$

Since the last condition is fulfilled, it can be concluded that the unit vector $\boldsymbol{t}$ can be uniquely determined if the functions K_S and K_N are given.

In fact, the equations introduced for the computation of the curvature functions of both the stream, and potential lines [1], form a system of two equations with the direction-vector field $\boldsymbol{t}$ being the unknown quantity:

$$\begin{aligned} K_S &= |\nabla \times \boldsymbol{t}|, \\ K_N &= \nabla \cdot \boldsymbol{t}. \end{aligned} \tag{2.25}$$

This system can be rewritten as

$$\boxed{\begin{aligned} K_S &= |\nabla \times (\cos\varphi, \sin\varphi)|, \\ K_N &= \nabla \cdot (\cos\varphi, \sin\varphi) \end{aligned}}, \tag{2.26}$$

hence forming a system of two equations with two unknown quantities, namely $\cos\varphi$ and $\sin\varphi$. Solving Eq. 2.26, is not trivial since there is no known closed-form solution. However, through the use of numerical methods, the flow direction can be uniquely determined. As a result, the velocity field is resolved everywhere in terms of both magnitude and direction if K_S and K_N are known functions.

Equation 1.10, which is the initial eikonal equation expressed in Cartesian coordinates, can generally not be solved analytically in terms of the velocity magnitude, v. However, expressing it in the streamline coordinate system leads to a solution for the velocity, justifying this coordinate-system selection. It must be noted that in Eq. 2.14 an integration of the potential line curvature K_N must be carried out along the streamlines of the field, while in Eq. 2.15 an integration of the streamline curvature must be carried out along the potential lines of the field. This 'crossing' relation is highly connected to the complementary character that potential flows possess. This particular property is of great geometric and physical importance and will be discussed in depth.

3 Geometric criterion for the determination of a Laplacian potential flow

By calculating the derivatives of Eqs. 2.10 and 2.11 with respect to n and s, respectively, it is obtained that

$$\frac{\partial K_N}{\partial n} = -\frac{\partial}{\partial n \partial s} \log v, \tag{3.1}$$

$$\frac{\partial K_S}{\partial s} = \frac{\partial}{\partial s \partial n} \log v. \tag{3.2}$$

From Eqs. 3.1 and 3.2 an interesting relation between the two curvatures, characterizing solely potential flows, can be derived:

$$(3.1) = -(3.2) \quad \Leftrightarrow \quad \boxed{\frac{\partial K_S}{\partial s} + \frac{\partial K_N}{\partial n} = 0}. \tag{3.3}$$

Equation (3.3) indicates that *at every point of a potential flow the sum of the rate of change of the streamline curvature, with respect to the streamline arclength, and the rate of change of the potential-line curvature, with respect to the potential-line arclength, equals zero.*

This result is a purely geometric criterion to represent the continuity and the irrotationality conditions for two-dimensional steady flows. More precisely, if the geometry of a set of flow trajectories, together with the geometry of the associated orthogonal trajectories, satisfies this equation then the flow field is irrotational and accordingly the velocity vector field possesses a potential. Inversely, an irrotational flow field will automatically satisfy Eq. 3.3.

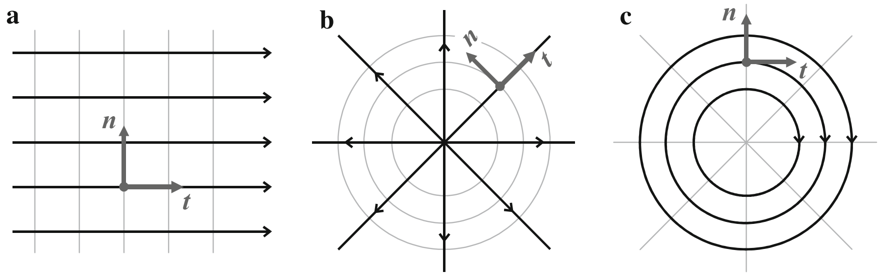


Fig. 3 Trajectories (black) and orthogonal trajectories (grey) for three sets of orthogonal curves

In other words, it is now obvious that the continuity and irrotationality conditions contain a geometric beauty, apart from being just assumptions simplifying the flow-physics analysis. Streamlines and potential lines appear to be equivalent and to have the same significance for potential flows. This kind of symmetry is depicted in Eq. 3.3 emphasizing their complementary character.

4 Applications

To further illustrate these ideas and their application to potential flows, they are applied to several sets of orthogonal curves. Initially, the use of Eq. 3.3 as a geometric criterion for the satisfaction of continuity and irrotationality is demonstrated for the following three families of orthogonal curves. The first set consists of parallel straight lines and the corresponding orthogonal trajectories, which are also parallel straight lines (Fig. 3a). In the second pattern the trajectories are radial lines originating from a common point and therefore the associated orthogonal trajectories are concentric circles having the same origin (Fig. 3b). The third set is the complementary case of the second family of orthogonal curves (Fig. 3c).

It can be very easily verified that Eq. 3.3 holds for all three cases. In the first case the two orthogonal sets of curves both have zero curvature functions, thus automatically satisfying the criterion. In the second case, Eq. 3.3 is again satisfied, since both sets of curves have constant curvatures, zero for the trajectories and non-zero for the orthogonal trajectories. Consequently, their derivatives with respect to their line elements are zero as well. Finally, in the last case the geometric criterion is satisfied given that the third set of orthogonal curves is complementary to the second. In conclusion, in all three cases the vector fields, which create the trajectories in Fig. 3, represent Laplacian potential flows. These sets correspond to the elementary flows in fluid mechanics: the uniform, the source and the vortex flow. The eikonal equation can now be solved to find their exact velocity distributions.

4.1 Uniform flow

The topology of the first case dictates that

$$K_S = 0, \quad K_N = 0.$$

Equation 2.18 gives:

$$v = C = \text{const},$$

which means that the velocity magnitude associated to the first topology must be constant everywhere in the flow field.

4.2 Source flow

From Fig. 3b it can be seen that

$$K_S = 0, \quad K_N = \frac{1}{s}$$

By substituting the functions above in Eq. 2.18 we obtain,

$$v = C\frac{1}{s} \quad \text{where } C = \mathrm{e}^c = \text{const.}$$

Thus, the velocity magnitude is inversely proportional to the streamline coordinate s, while it remains constant along a potential line. By replacing the variable s with $\sqrt{x^2+y^2}$ and the constant C with $E/(2\pi)$ we obtain the general formula for the velocity distribution of a source in Cartesian coordinates, where E is the source strength,

$$v = \frac{E}{2\pi}\frac{1}{\sqrt{x^2+y^2}}.$$

4.3 Vortex flow

From Fig. 3c it can be seen that

$$K_S = -\frac{1}{n}, \quad K_N = 0.$$

It is noteworthy that in this particular case the streamline curvature takes negative values everywhere due to the clockwise sense of rotation of the unit tangent vector. Eq. 2.18 yields:

$$v = C\frac{1}{n} \quad \text{where } C = \mathrm{e}^c = \text{const.}$$

Thus, the velocity magnitude is inversely proportional to the coordinate n while it remains constant along a streamline. By replacing the variable n with $\sqrt{x^2+y^2}$ and the constant C with $\Gamma/(2\pi)$ we obtain the general formula for the velocity distribution of a vortex in Cartesian coordinates with circulation Γ:

$$v = \frac{\Gamma}{2\pi}\frac{1}{\sqrt{x^2+y^2}}.$$

The same result can be obtained by solving Eq. 1.10 directly using Cartesian coordinates. This can be performed since Eq. 1.10 has an analytical solution for this particular case. More precisely, the solution to the eikonal equation

$$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 = f(x^2+y^2),$$

which can be interpreted as the Hamiltonian equation, describing the planar motion of a point with unit mass caused by the presence of a central force, was found to be the following [4]:

$$F = -D\arctan\frac{x}{y} \pm \frac{1}{2}\int\sqrt{\mathrm{e}^\rho f(\mathrm{e}^\rho) - D^2}\mathrm{d}\rho + E,$$

where $\mathrm{e}^\rho = r^2 = x^2+y^2$ and $D = x\dfrac{\partial F}{\partial y} - y\dfrac{\partial F}{\partial x}$.

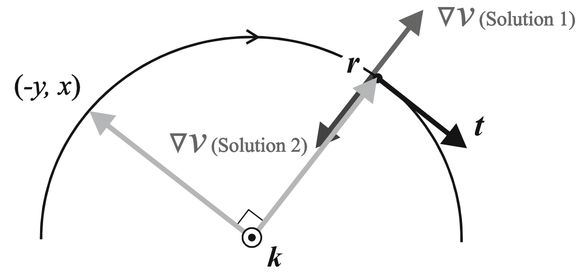
In this case, the global curvature of the vortex flow presented in Fig. 3c is

$$K_G^2(x,y) = K_S^2 + K_N^2 = \left(-\frac{1}{\sqrt{x^2+y^2}}\right)^2 + 0 = \frac{1}{x^2+y^2}.$$

Thus, according to [4], the eikonal equation

$$\left(\frac{\partial N}{\partial x}\right)^2 + \left(\frac{\partial N}{\partial y}\right)^2 = \frac{1}{x^2+y^2} = f(x^2+y^2),$$

Fig. 4 Vortex flow



has the following solution:

$$N = -D \arctan\frac{x}{y} \pm \frac{1}{2}\int \sqrt{e^{\rho} f(e^{\rho}) - D^2}\, d\rho + E \Rightarrow$$
$$N = -D \arctan\frac{x}{y} \pm \frac{1}{2}\int \sqrt{1 - D^2}\, d\rho + E \Rightarrow$$
$$N = -D \arctan\frac{x}{y} \pm \frac{1}{2}\left(\sqrt{1 - D^2}\right)\rho + E \Rightarrow$$
$$N = -D \arctan\frac{x}{y} \pm \frac{1}{2}\left(\sqrt{1 - D^2}\right)\log\left(x^2 + y^2\right) + E.$$

From the characteristic equation we have:

$$D = x\frac{\partial N}{\partial y} - y\frac{\partial N}{\partial x} \Rightarrow D = \nabla N\cdot(-y, x) \overset{N = \log \nu}{\Rightarrow} D = \frac{1}{\nu}\nabla\nu\cdot(-y, x).$$

Using Eq. 1.5, which is the direct result of the conservation of mass and taking into account that $K_N = 0$ we may derive

$$(1.5) \overset{K_N=0}{\Rightarrow} \nabla\nu\cdot\boldsymbol{t} = 0 \Rightarrow \nabla\nu\perp\boldsymbol{t}.$$

If $\boldsymbol{r} = (x, y)$ represents the position vector, the geometry of the problem always implies that $\boldsymbol{r}\perp\boldsymbol{t}$ (Fig. 4). Since $\boldsymbol{r}\perp(-y, x)$ it is concluded that $\nabla\nu\perp(-y, x)$, which in turn means that $D = 0$ and thus the solution is simplified to:

$$N = \pm\frac{1}{2}\log\left(x^2 + y^2\right) + E \overset{C = e^E}{\Rightarrow} N = \pm\log\sqrt{x^2 + y^2} + \log C \Rightarrow \begin{cases} N = \log\left(C\sqrt{x^2 + y^2}\right) \\ N = \log\dfrac{C}{\sqrt{x^2 + y^2}} \end{cases}$$

or

$$\begin{cases} \nu = C\sqrt{x^2 + y^2} & \text{Solution 1} \\ \nu = \dfrac{C}{\sqrt{x^2 + y^2}} & \text{Solution 2} \end{cases}$$

The velocity gradient corresponding to Solution 1 is:

$$\nabla\nu = \left(\frac{C}{\sqrt{x^2 + y^2}}\right)\boldsymbol{r}.$$

Since the constant C is positive by definition ($C = e^E$) it is concluded that the velocity gradient is pointing outwards (Fig. 4). On the other hand, Eq. 1.6, which is an alternative expression for the irrotationality condition, implies that:

$$\nu K_S \boldsymbol{k} = \boldsymbol{t} \times \nabla\nu \overset{K_S<0}{\Rightarrow} \nabla\nu \text{ has the opposite direction of } \boldsymbol{r},$$

that is, the velocity gradient must be pointing inwards. Hence, by making use of both the continuity and irrotationality conditions, Solution 1 is rejected as physically impossible. The same result could be deduced from the geometrical characteristic of potential flows developed in Appendix A, upon which ∇v *always lies in the overlapping area formed by the concave parts of the streamlines and potential lines* (Fig. 7).

On the other hand, Solution 2 represents the velocity magnitude of the potential flow depicted in Fig. 4 since the associated velocity gradient is pointing inwards

$$\nabla v = \left(\frac{-C}{\left(x^2 + y^2\right)^{3/2}}\right)\boldsymbol{r},$$

thus satisfying the irrotationality condition. Therefore, replacing the constant C with $\Gamma/2\pi$ yields the same formula for the velocity distribution of a vortex:

$$v = \frac{\Gamma}{2\pi}\frac{1}{\sqrt{x^2 + y^2}}.$$

It is concluded that the streamline coordinate system proves to be advantageous over the Cartesian. Its application is not limited to specific cases and provides relatively quickly the analytical expression of the velocity magnitude.

It is noted that the velocity distributions of all three elementary flows were derived in a very intuitive manner, based on the overall 'flow picture', without the need of solving the Laplace equation. Not only is a simple observation of the flow field adequate to verify whether or not the continuity and irrotationality conditions are fulfilled, but also in case they are, it is sufficient for the derivation of its velocity distribution. As a matter of fact, the eikonal equation replaces the Laplace equation with the aid of the flow topology.

Considering the above, flow visualization plays a very important role, not only in revealing the basic flow features but also in offering the possibility to obtain velocity and pressure data. From an experimental point of view this can be achieved by first generating a 'flow picture' using a smoke, dye or hydrogen bubble visualization technique [5,6]. The next step would be to photogrammetrically digitalize and eventually reconstruct this flow picture in silico [7]. Finally, this reconstruction could be post-processed, according to the presented technique, to provide the velocity and the pressure properties throughout the flow field.

5 The Global Curvature vector and its 'Curvature Potential'

Equations 2.10 and 2.11 show that for a Laplacian potential flow the derivatives of the function $\log v$ with respect to s and n give the curvatures $-K_N$ and K_S, respectively. Therefore, the scalar serves as the potential function of a vector field, the components of which are the streamline and potential-line curvatures. Because the magnitude of this vector is equal to the global curvature $K_G(K_G = \sqrt{K_S^2 + K_N^2})$, it is reasonable to name it the *Global Curvature Vector*, $\boldsymbol{K_G}$. Consequently it can be written as follows:

$$\nabla v = v\boldsymbol{K_G} \quad \text{or} \quad \boxed{\boldsymbol{K_G} = \nabla N} \quad \text{where} \quad \begin{cases} \boldsymbol{K_G} = (-K_N, K_S) \\ N = \log v \end{cases}. \tag{5.1}$$

It is noteworthy that the Global Curvature vector is parallel to the velocity gradient ∇v (Fig. 5) and its magnitude is v-times smaller than ∇v.

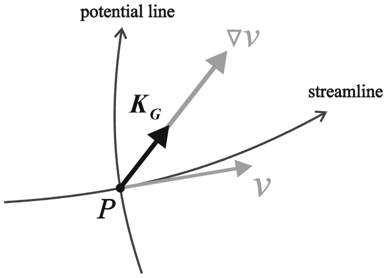


Fig. 5 The Global Curvature vector $\boldsymbol{K_G}$ of a potential flow

Table 1 Overview of the developed methods for the computation of the curvature functions for the sets of curves that form an orthogonal net

Method	Trajectory (streamline)	Orthogonal trajectory
Directional derivative (1)	$K_S(x, y) = \nabla\varphi \cdot \boldsymbol{t}$	$K_N(x, y) = \nabla\left(\varphi + \frac{\pi}{2}\right) \cdot \boldsymbol{n} = \nabla\varphi \cdot \boldsymbol{n}$
Rotation (2)	$K_S(x, y) = \lvert\nabla \times \boldsymbol{t}\rvert$	$K_N(x, y) = \lvert\nabla \times \boldsymbol{n}\rvert = \nabla \cdot \boldsymbol{t}$
Curvature potential N (3)	$K_S(s, n) = \dfrac{\partial N}{\partial n}$	$K_N(s, n) = -\dfrac{\partial N}{\partial s}$

This is of great interest since the global curvature vector $\boldsymbol{K_G}$ *gives both the direction and the magnitude of the maximum spatial percentage rate of increase of the velocity*. Considering the fact that in a potential flow ∇v is collinear with the fluid-flow acceleration, it is concluded that the global curvature vector $\boldsymbol{K_G}$ is collinear with the fluid-flow acceleration as well. Therefore it has a strong physical meaning.

The geometric formulation of the potential theory is depicted in Eq 5.1 which is the vectorial expression of the eikonal equation (1.7). In traditional potential theory, the velocity components can be expressed in terms of a single variable Φ such that:

$$\boldsymbol{v} = \nabla\Phi, \text{ or equivalently } v_x = \frac{\partial\Phi}{\partial x} = \Phi_x \text{ and } v_y = \frac{\partial\Phi}{\partial y} = \Phi_y. \tag{5.2}$$

Likewise, the Global Curvature vector components, can be expressed in terms of a single variable N, which is the natural logarithm of the flow-field velocity, $\log v$:

$$\boldsymbol{K_G} = \nabla N, \text{ or equivalently } K_N = -\frac{\partial N}{\partial s} \text{ and } K_S = \frac{\partial N}{\partial n}. \tag{5.3}$$

The method proposed above for the computation of the streamline and potential-line curvature is referred to as the *'Method of Curvature Potential'*. Equations (5.3) automatically satisfy the geometric criterion for continuity and irrotationality (Eq. 3.3). As a result, the existence of the analytic expression N, which satisfies $\boldsymbol{K_G} = \nabla N$, is an alternative for the validity of these two kinematical conditions.

In conclusion, there are three methods for the calculation of the curvature functions for the sets of curves that create an orthogonal net: the method of directional derivative, the method of rotation and the newly introduced method of curvature potential. All available techniques are outlined in table 1. The streamline and orthogonal trajectory curvatures can be computed at every point, as long as the angle function, φ and/or the velocity vector field, $\boldsymbol{v}$, or the velocity magnitude, v, is given. Topological information can thus be extracted, even without knowing the analytic expressions of the streamlines and the associated orthogonal trajectories. It is important to mention that continuity and irrotationality need not be satisfied for the application of the first two methods. However, the third method must fulfill these two conditions, since it is obtained for potential flows only. In this case the term 'orthogonal trajectory' can be replaced by the term 'potential line'.

6 The Global Curvature vector and its mathematical meaning

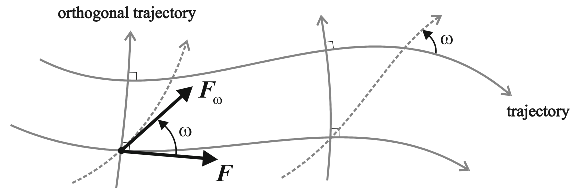
Given a vector field $\boldsymbol{F}$, not necessarily irrotational, we can produce a new vector field $\boldsymbol{F}_\omega$ for which the direction of $\boldsymbol{F}$ is rotated counterclockwise by ω at every point, while the magnitude remains unchanged [8]; $\boldsymbol{F}_\omega$ is called *the rotated vector field of* $\boldsymbol{F}$ by ω (Fig. 6). If $K_{S,\omega}$ represents the streamline curvature of the rotated vector field $\boldsymbol{F}_\omega$, it can be proven [8] that for every angle ω:

$$K_{S,\omega} = K_S \cos\omega + K_N \sin\omega, \tag{6.1}$$

where K_S and K_N are the trajectory and orthogonal trajectory curvature functions of the initial vector field $\boldsymbol{F}$. Additionally, it is shown in [8] that the extreme values of $K_{S,\omega}$ for all angles ω are:

$$K_{S,\omega}\big|_{\text{MAX}} = \sqrt{K_S^2 + K_N^2}, \quad K_{S,\omega}\big|_{\text{MIN}} = -\,K_{S,\omega}\big|_{\text{MAX}}. \tag{6.2}$$

Fig. 6 Initial vector field $\boldsymbol{F}$, which has undergone a counterclockwise rotation by ω at every point



In other words, *the magnitude of the global curvature vector,* $\boldsymbol{K_G}$*, of the initial vector field* $\boldsymbol{F}$ *always represents the extreme values of the streamline curvature* $K_{S,\omega}$ *of the rotated vector field* $\boldsymbol{F}_\omega$, that is:

$$\boxed{K_{S,\omega}\big|_{\mathrm{MAX}} = K_G, \quad K_{S,\omega}\big|_{\mathrm{MIN}} = -K_G}. \tag{6.3}$$

Furthermore, a theorem regarding the uniqueness of curvature description for vector fields is presented in [8]. In particular, given two vector fields $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ which have non-constant direction fields and for which $K_{S,1} = K_{S,2}$ and $K_{N,1} = K_{N,2}$, that is, their trajectory and orthogonal trajectory curvature functions are identical and the directions of the vectors of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ coincide in every point. Taking into account the definition of the global curvature vector (Eq. 3.3), this theorem can be updated as follows:

Given are two vector fields $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ *with non-constant direction fields. If their global curvature vectors are equal,* $\boldsymbol{K}_{G,1} = \boldsymbol{K}_{G,2}$*, then the directions of* $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ *coincide at every point.*

Knowledge of the trajectory and the orthogonal-trajectory curvature functions provides all information about the topology of the vector field. This guarantees the uniqueness of the solution for the direction of the velocity vector field $\boldsymbol{t}$ which can be derived after solving the system of Eqs. (2.26).

7 Conclusions

The main purpose of this research has been to combine the kinematics of a two-dimensional incompressible steady flow with its topology, thus giving a geometrical interpretation to traditional potential theory. Imposing two kinematical conditions (continuity and irrotationality) and applying the mathematical method developed for the flow geometric representation (method of rotation, [1]) an eikonal equation was obtained (Eq. 1.10 or Eq. 5.1). This has been solved analytically so that the flow-field velocity can be calculated if the streamline and potential-line curvature functions are known Eq. 2.18. Then, by substituting the velocity in the Bernoulli equation, the pressure-field properties can be computed directly. Hence, it can be concluded that *visualizing a potential flow is equivalent to measuring its velocity and pressure-field properties given the required boundary conditions*.

Several flow-visualization techniques are available for the extraction of topological information contained in data sets, such as those obtained in computer-simulated flows. The Hodge decomposition [9] for instance provides one of the best tools for this purpose. It allows the detection of critical points and the visual deduction of the entire vector-field structure. Nevertheless, this decomposition, as well as other flow-visualization tools, differs from the geometric potential technique presented in this paper. The latter follows exactly the opposite line of thought, since it allows new physical insights to be derived from the flow topology.

Moreover, an exclusively geometric criterion was developed, capable of determining whether a flow field possesses a potential. It points out that if the sum of the streamline and its orthogonal trajectory curvature rates of change with respect to their arc lengths equals zero (Eq. 3.3), then the flow satisfies both the continuity and irrotationality conditions and vice versa. In fact, one can draw conclusions regarding irrotationality based exclusively on a visualization picture of the flow streamlines.

Furthermore, the streamline and potential-line curvature functions were combined to form a new vector, the global curvature vector $\boldsymbol{K_G}$ (Eq. 5.1), whose magnitude is exactly the scalar function K_G introduced in [1]. Particularly for potential flows a physical interpretation was found: $\boldsymbol{K_G}$ *is collinear with the fluid-flow acceleration and gives both the direction and the magnitude of the maximum spatial percentage rate of increase of* v. Apparently,

$\boldsymbol{K_G}$ loses its 'significance' when it is defined for non-potential flows. For this general case, only the magnitude of $\boldsymbol{K_G}$ is still important by representing the extreme values of the streamline curvature $K_{S,\omega}$ of the rotated flow field (Eq. 6.3).

Finally, it has been shown that there exists a potential function for finding the components of the global curvature vector. This function is the natural logarithm of the velocity magnitude $\log v$; it was given the symbol N and the name curvature potential. It was proven that continuity and irrotationality are identically satisfied if $\boldsymbol{K_G} = \nabla N$. In other words: the existence of such an analytic expression N is the necessary and sufficient condition for $\boldsymbol{v}$ to represent a Laplacian velocity field.

Although the mathematical findings and the conclusions drawn in this work were obtained while working on fluids, it is worthwhile mentioning that they could be automatically generalized to apply to any two-dimensional steady vector field.

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Appendix A: Geometrical characteristic of two-dimensional potential flows

An interesting geometric feature inherent to potential flows is presented here. Eqs. 1.5 and 1.6 could be further simplified by developing their dot and cross products, respectively. This requires some knowledge of the flow-field topology: as the streamline and potential-line curvatures can take either positive or negative values (signed curvature) at a given point, the angle ϑ, between the velocity vector $\boldsymbol{v}$ and the velocity gradient ∇v, should belong to a specific quadrant so that the equality of Eq. 1.6 and Eq. 1.5 will be assured simultaneously. Following the right-hand rule for the curvature sign and supposing that ϑ is always measured from the velocity vector $\boldsymbol{v}$ anti-clockwise, the analysis for all possible flow topologies that might be encountered at a point (Fig. 7) is carried out:

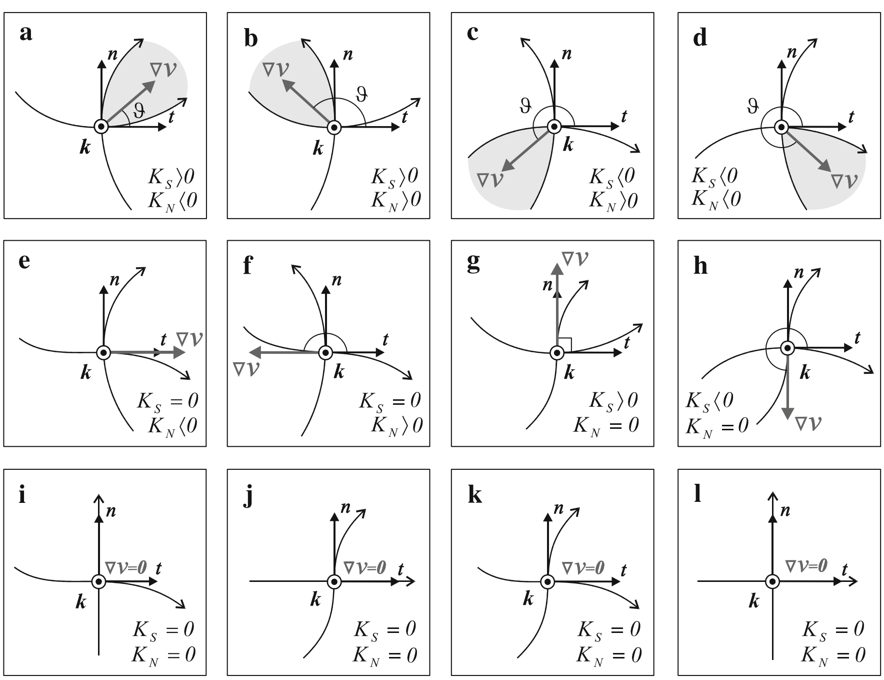

Fig. 7 Schematic representation of all possible flow topologies based on the signs of the streamline and potential-line curvatures

Case a If K_S is positive, it is concluded from Eq. 1.6 that the cross-product $\boldsymbol{t} \times \nabla \nu$ must be parallel to $\boldsymbol{k}$, which is the unit vector perpendicular to the plane of the fluid motion pointing out of the page. This in turn means that the angle ϑ, between $\boldsymbol{\nu}$ and $\nabla \nu$ must give a positive $\sin \vartheta$, indicating that $\vartheta \ni (0, \pi)$. In addition, K_N has a negative value; thus, from Eq. 1.5, $\cos \vartheta$ must be positive or $\vartheta \ni \left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$. In order to satisfy both equations, the angle ϑ must finally belong to the first quadrant, $\vartheta \ni \left(0, \frac{\pi}{2}\right)$. This explains why in Fig. 1a the $\nabla \nu$ lies somewhere between the streamwise and normal directions, $\boldsymbol{t}$ and $\boldsymbol{n}$. Eqs. 1.5 and 1.6 can then be written as

$$(1.5) \overset{\vartheta \ni (0, \frac{\pi}{2})}{\Rightarrow} \nu K_N = -|\nabla \nu| \cos \vartheta \quad \text{and} \tag{A.1}$$

$$(1.6) \overset{\vartheta \ni (0, \frac{\pi}{2})}{\Rightarrow} \nu K_S = +|\nabla \nu| \sin \vartheta. \tag{A.2}$$

Following the same line of thought for the remaining cases we obtain the following results:

Case b

$$\left.\begin{array}{l} K_N \rangle 0 \overset{(1.5)}{\Rightarrow} \vartheta \ni \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ K_S \rangle 0 \overset{(1.6)}{\Rightarrow} \vartheta \ni (0, \pi) \end{array}\right\} \Rightarrow \vartheta \ni \left(\frac{\pi}{2}, \pi\right). \text{ Therefore, } \left\{\begin{array}{l} (1.5) \Rightarrow \nu K_N = -|\nabla \nu| \cos \vartheta \\ (1.6) \Rightarrow \nu K_S = +|\nabla \nu| \sin \vartheta \end{array}\right. ;$$

Case c

$$\left.\begin{array}{l} K_N \rangle 0 \overset{(1.5)}{\Rightarrow} \vartheta \ni \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ K_S \langle 0 \overset{(1.6)}{\Rightarrow} \vartheta \ni (\pi, 2\pi) \end{array}\right\} \Rightarrow \vartheta \ni \left(\pi, \frac{3\pi}{2}\right). \text{ Therefore, } \left\{\begin{array}{l} (1.5) \Rightarrow \nu K_N = -|\nabla \nu| \cos \vartheta \\ (1.6) \Rightarrow \nu K_S = +|\nabla \nu| \sin \vartheta \end{array}\right. ;$$

Case d

$$\left.\begin{array}{l} K_N \langle 0 \overset{(1.5)}{\Rightarrow} \vartheta \ni \left(\frac{3\pi}{2}, \frac{\pi}{2}\right) \\ K_S \langle 0 \overset{(1.6)}{\Rightarrow} \vartheta \ni (\pi, 2\pi) \end{array}\right\} \Rightarrow \vartheta \ni \left(\frac{3\pi}{2}, 2\pi\right). \text{ Therefore, } \left\{\begin{array}{l} (1.5) \Rightarrow \nu K_N = -|\nabla \nu| \cos \vartheta \\ (1.6) \Rightarrow \nu K_S = +|\nabla \nu| \sin \vartheta \end{array}\right. ;$$

Case e

$$\left.\begin{array}{l} K_N \langle 0 \overset{(1.5)}{\Rightarrow} \vartheta \ni \left(\frac{3\pi}{2}, \frac{\pi}{2}\right) \\ K_S = 0 \overset{(1.6)}{\Rightarrow} \vartheta = 0, \quad \vartheta = \pi \text{ or } |\nabla \nu| = 0 \end{array}\right\} \Rightarrow \vartheta = 0. \text{ Therefore, } \left\{\begin{array}{l} (1.5) \Rightarrow \nu K_N = -|\nabla \nu| \cos \vartheta \\ (1.6) \Rightarrow \nu K_S = +|\nabla \nu| \sin \vartheta \end{array}\right. ;$$

Case f

$$\left.\begin{array}{l} K_N \rangle 0 \overset{(1.5)}{\Rightarrow} \vartheta \ni \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ K_S = 0 \overset{(1.6)}{\Rightarrow} \vartheta = 0, \quad \vartheta = \pi \text{ or } |\nabla \nu| = 0 \end{array}\right\} \Rightarrow \vartheta = \pi. \text{ Therefore, } \left\{\begin{array}{l} (1.5) \Rightarrow \nu K_N = -|\nabla \nu| \cos \vartheta \\ (1.6) \Rightarrow \nu K_S = +|\nabla \nu| \sin \vartheta \end{array}\right. ;$$

Case g

$$\left.\begin{array}{l} K_N = 0 \overset{(1.5)}{\Rightarrow} \vartheta = \frac{\pi}{2}, \vartheta = \frac{3\pi}{2} \text{ or } |\nabla \nu| = 0 \\ K_S \rangle 0 \overset{(1.6)}{\Rightarrow} \vartheta \ni (0, \pi) \end{array}\right\} \Rightarrow \vartheta = \frac{\pi}{2}. \text{ Therefore, } \left\{\begin{array}{l} (1.5) \Rightarrow \nu K_N = -|\nabla \nu| \cos \vartheta \\ (1.6) \Rightarrow \nu K_S = +|\nabla \nu| \sin \vartheta \end{array}\right. ;$$

Case h

$$\left.\begin{array}{l} K_N = 0 \overset{(1.5)}{\Rightarrow} \vartheta = \frac{\pi}{2}, \vartheta = \frac{3\pi}{2} \text{ or } |\nabla \nu| = 0 \\ K_S \langle 0 \overset{(1.6)}{\Rightarrow} \vartheta \ni (\pi, 2\pi) \end{array}\right\} \Rightarrow \vartheta = \frac{3\pi}{2}. \text{ Therefore, } \left\{\begin{array}{l} (1.5) \Rightarrow \nu K_N = -|\nabla \nu| \cos \vartheta \\ (1.6) \Rightarrow \nu K_S = +|\nabla \nu| \sin \vartheta \end{array}\right. ;$$

Cases i, j, k and l For the simple cases where both curvatures vanish, Eqs. 1.5 and 1.6 imply that the velocity gradient must also vanish:

$$\left.\begin{array}{l} K_N = 0 \overset{(1.5)}{\Rightarrow} \vartheta = 0 \text{ or } |\nabla \nu| = 0 \\ K_S = 0 \overset{(1.6)}{\Rightarrow} \vartheta = \frac{\pi}{2} \text{ or } |\nabla \nu| = 0 \end{array}\right\} \Rightarrow |\nabla \nu| = 0. \text{ Therefore, } \left\{\begin{array}{l} (1.5) \Rightarrow \nu K_N = -|\nabla \nu| \cos \vartheta \\ (1.6) \Rightarrow \nu K_S = +|\nabla \nu| \sin \vartheta \end{array}\right. .$$

It was confirmed that for every case Eqs. 1.5 and 1.6 are automatically satisfied only if the angle ϑ is such that *the velocity gradient* $\nabla \nu$ *always lies in the overlapping area formed by the concave parts of the streamlines and potential lines* (Fig. 7). In other words, its orientation on the plane is related to the flow topology. That is an additional indication that the irrotationality, together with the continuity condition, implies a geometric constraint on two-dimensional steady flows.

Concluding, it has been shown that Eqs. 1.5 and 1.6 which hold only for potential flows can always be reduced to Eqs. A.1 and A.2, respectively, independent of the flow topology. If now the later equations are squared and subsequently added together, they give the eikonal equation (Eq. 1.10) expressed in Cartesian coordinates (x, y).

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